Thresholds for LDPC codes over OFDM

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Background on LDPC codes

- Low Density Parity Check (LDPC) codes
  - linear codes with sparse parity-check matrices
  - simple definition, but celebrated capacity-approaching performance
- LDPC versus Classical codes
  - LDPC: large ensembles of codes - all with same performance
  - random code from ensemble performs close to average
    - analysis: performance averaged over ensemble
  - design: ensemble with good average performance
    - code randomly chosen from ensemble for actual use
- How is the ensemble specified?
  - weights of the columns and rows of the parity-check matrix
  - weights are collected into weight distribution polynomials

Analysis and Design Tools for LDPC Codes

Study average performance of ensemble of codes whose parity-check matrices have the same weight distribution

Message-Passing Decoders, Thresholds, Density Evolution

- Message-passing decoders: practical, iterative, capacity-approaching
  - performance of ensemble is studied under message-passing decoding
- Average performance of ensemble shows a threshold phenomenon
  - threshold = SNR* $\rightarrow$ SNR $>$ SNR* will result in successful decoding
  - block-length $\rightarrow$ infinity, iterations $\rightarrow$ infinity
  - threshold is a function of weight distribution
- Density evolution
  - tool to determine the average performance of an ensemble
  - at a particular SNR, after a particular iteration for block-length $\rightarrow$ infinity
  - can be used to determine threshold

Study of LDPC codes in a new system involves...

developing a density evolution algorithm and determination of threshold

OFDM

System Descrption

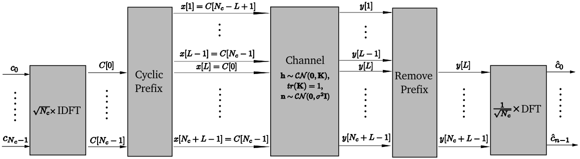


Figure: OFDM System.

- Assumption: Transmission over an ISI channel with with L fixed taps.
- The channel is modeled as

$$\hat{\mathbf{c}} = \mathbf{H}.\mathbf{c} + \mathbf{N}$$

- $\mathbf{c}$ is the input vector, $\hat{\mathbf{c}}$ the output vector, $\mathbf{H}$ the DFT of the *Channel Impulse Response* and $\mathbf{N}$ the normalized DFT of the random noise vector.
- Binary Input alphabet.BPSK modulation.

LDPC over OFDM

- Assumption
  - A codeword is distributed over a single OFDM symbol.
  - The blocklength of the code (N_c) tend to infinity.
- In the limit the cyclic prefix gives no overhead.

Prior Work on LDPC Codes in an OFDM System

- Prior work on LDPC over OFDM/ISI
  - Manoni *et al*: mixture PDF analysis and optimization of degree distribution
  - Aazhang *et al*: positioning of information bits in OFDM subcarriers
  - Kavcic *et al*: LDPC codes over binary-input ISI channels with BCJR
  - LDPC coding for OFDM has been addressed by many researchers by simulations
- Previous theoretical works employ a Gaussian approximation for threshold estimation
- No rigorous proof for the existence of threshold in OFDM systems

Our Work

- Propose a rigorous density evolution.
- Existence of LDPC thresholds.
- Method for threshold estimation.
- Comparisons between the time-domain BCJR algorithm proposed by Kavciv *et al*

Tanner graph representation of LDPC codes

- Tanner graph : A bipartite graph.
  - Variable nodes.
  - Check nodes.
- Check node j is connected to the variable node i whenever the element h_ij in H is 1.

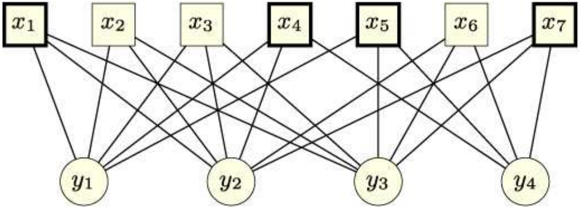


Figure: Tanner Graph

LDPC Code Description

- Irregular LDPC codes are specified by the degree distribution polynomials.
- Variable node degree distribution
  $\lambda(x) = \sum_{i=2}^{d_v} \lambda_i x^{i-1}$
  λ_i denotes the fraction of all edges connected to variable nodes of degree i and d_v is the maximum variable node degree.
- Check node degree distribution
  $\rho(x) = \sum_{j=2}^{d_c} \rho_j x^{j-1}$, ρ_j
  ρ_j denotes the fraction of all edges connected to check nodes of degree j and d_c is the maximum check node degree.
- The triplet (λ, ρ, n) specifies an ensemble of LDPC codes.

Message Passing

- A set of Iterative algorithms.
- Passing messages on the edges of a graph iteratively.
- Only extrinsic information is passed along.
- Update the estimate of the codeword bits in each iteration.
- A pair of update equation.
    - Variable node update (Ψ).
    - Check node update (Φ)

Density Evolution: Steps

- Check symmetry conditions.
- Calculate Probability Distribution Function (pdf) of a message in a random edge.
    - Pr(message in a random edge is in error).
- Evolve this pdf through iteration.
- Check if the Pr(message in a random edge is in error) $\rightarrow 0$ as the number of iteration $\rightarrow \infty$.
    - Threshold Estimation.

Symmetry Conditions

- Check Node Symmetry
- Variable Node Symmetry
- Channel Symmetry.

$$p(y_t = q|x_t = 1) = p(y_t = -q|x_t = -1)$$

- Symmetry of distribution.
- Conditional Independence of error probability under symmetry.

Symmetry in OFDM

Channel Symmetry

- OFDM channel is a *Symmetric Channel*
  $p_{Z_i|C_i}(z_i|c_i = 1) = p_{Z_i|C_i}(-z_i|c_i = -1)$
- The OFDM channel LLR is defined as: $u_i = L(z_i) := \ln \left[\frac{p_{Z_i|C_i}(z_i|c_i=1)}{p_{Z_i|C_i}(z_i|c_i=-1)} \right]$
  $U_i \sim \mathcal{N}\left(\frac{4|H[i]|^2}{\sigma^2}, \frac{8|H[i]|^2}{\sigma^2} \right)$.
- LLR distribution is symmetric.

$$f_{U_i}(u_i) = \exp(u_i).f_{U_i}(-u_i)$$

- Analysis can be restricted to the All-one Codeword.

Initial PDF estimation and Density Evolution Algorithm

The algorithm

Consider a degree distribution pair (λ, ρ) *and transmission over an OFDM channel with* N_c *subcarriers with code of blocklength* $n = N_c$*, with associated L-densities* $\tilde{f}_i, i \in \{1, 2, \ldots, N_c\}$ *. Define*

$$f_0 = \frac{1}{N_c} \sum_{i=1}^{N_c} \tilde{f}_i,$$

then for $l \geq 1$*,*

$$f_l = f_0 \otimes \lambda\left(\rho\left(f_{l-1}\right)\right),$$

where for L-density f

$$\lambda(f) := \sum_i \lambda_i f^{\otimes(i-1)}, \qquad \rho(f) := \sum_i \rho_i f^{\boxtimes(i-1)}.$$

Monotonicity and Threshold

- The update equations : Same as AWGN.
- Same monotonicity argument.
- Existence of threshold !!

Threshold Estimation

- We let the number of subcarriers N_c tend to infinity.
- LLR distribution depends on the the DTFT of the channel impulse response $(H(e^{j\omega}))$
- LLR distribution is now a continuous function ofthe angular frequency ω

$$f(u,\omega) = \frac{\sigma}{4|H(e^{j\omega})|\sqrt{\pi}} \exp\left[-\frac{(\sigma^2 u - 4|H(e^{j\omega})|^2)^2}{16|H(e^{j\omega})|^2\sigma^2}\right]$$

$$H(e^{j\omega}) = \sum_{i=-\infty}^{\infty} h[i]e^{-j\omega_i}$$

- Summation in the equation changes to an integeral

$$f_0(u) = \frac{1}{2\pi}\int_0^{2\pi} f(u,\omega).d\omega$$

Threshold Estimation with Spectral Nulls

- The function $f(u,\omega)$ is not always well behaved.
- Problems in channels with spectral nulls.
- New approach to calculate the $f_0(u)$.
- Using the idea of characteristic function

$$\begin{array}{ccc} f(u,\omega) & \nrightarrow & f_0(u) \\ \downarrow & & \uparrow \\ \hat{f}(t,\omega) & \rightarrow & \hat{f}(t) \end{array}$$

Threshold Estimation

$$\begin{aligned}\hat{f}(t,\omega) &:= \int_{-\infty}^{\infty} f(u,\omega)e^{jut}du \\ &= \exp\left[-\frac{4|H(e^{j\omega})|^2 t^2}{\sigma^2} + j\frac{4|H(e^{j\omega})|^2 t}{\sigma^2}\right] \\ \hat{f}(t) &:= \frac{1}{2\pi}\int_0^{2\pi} \hat{f}(t,\omega)\ d\omega \\ f_0(u) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(t)\ e^{-jut}\ dt\end{aligned}$$

- Advantage: A well behaved characteristic function obtained analytically.

Results

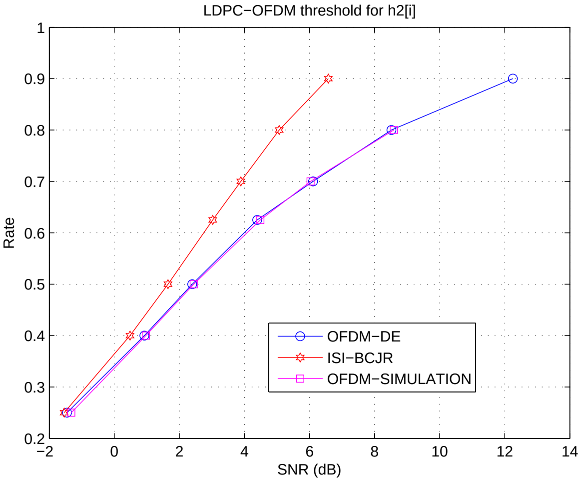
- Thresholds obtained for different rate regular LDPC codes.
- Comparison with LDPC BCJR threshold over a binary ISI channel.

Results

- Channel without spectral null

.

$$\{h_2[i]\} = 0.800, 0.600$$



Results

- Channel with spectral null

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$$\{h_1[i]\} = \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}.$$

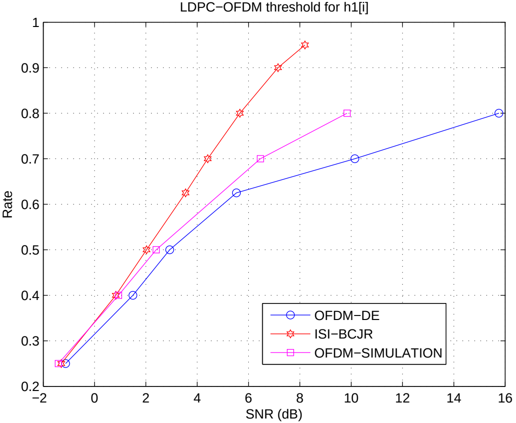

Figure: OFDM and ISI Thresholds against rate for $\{h_1[i]\}$

Conclusions

- Developed rigorous density evolution for binary-input OFDM
- Estimation of threshold - some trouble with spectral nulls
- Future work
  - Optimization of degree distributions for OFDM
  - Accurate estimation of thresholds with spectral nulls
  - Extension to wireless channels